数理解析研究所講究録
1123 巻 2000 年 43-51

Structure of solutions to the equation $\Delta u+u^{p}=0$ near a singular radial solution

Futoshi Takahashi 高橋 太
Department of mathematics, Faculty of Science,
Tokyo Institute of Technology

1 Introduction

Here we study the structure of the set of solutions to some nonlinear elliptic partial differential equation in a punctured ball. These solutions may be singular at the origin and satisfy the equation in the distribution sense.

Our main result is that, when measured by a weighted Hölder norm with a suitably chosen weight parameter, near an explicit radial solution that is singular at the origin, the set of these solutions is a smooth Banach manifold.

R.Hardt and L.Mou studied in [HM] the structure of the space of harmonic maps near a singular homogeneous harmonic map and proved that this is a smooth Banach manifold. We follow with modifications some steps in their study.

Let $\mathbf{B}_{r}=B_{r}^{n}(0)$ be a closed ball of radius r $(n\geq 3)$, and let $\mathbf{B}_{0}=\mathbf{B}_{1}\setminus\{0\}$ be a punctured ball.

For $p>\frac{n}{n-2}$, we consider the following equation

$$(1)\qquad \begin{cases}\Delta u+u^{p}=0 & \text{in } \mathbf{B}_{0},\\ u\in C^{2}(\mathbf{B}_{0}), \quad u\geq 0 & \text{in } \mathbf{B}_{0}.\end{cases}$$

We recall two facts about the equation (1):

(a) [PL] When $p\geq\frac{n}{n-2}$, any solution of (1) satisfies

$$\begin{cases}\Delta u+u^{p}=0 & \text{in } \mathcal{D}'(\mathbf{B}_{1}),\\ u\in L^{p}(\mathbf{B}_{1}),\end{cases}$$

that is, all solutions of (1) extend to the whole ball as solutions in the distribution sense.

(b) [GS][CGS] When $\frac{n}{n-2}<p\leq\frac{n+2}{n-2}$, any solution of (1) satisfies

$$u(x)\leq C|x|^{\frac{-2}{p-1}}$$

near $x=0$ for some constant $C>0$.

Taking account of these facts, we consider the following set of solutions of (1) with a specific growth rate at the origin:

$$(2)\qquad \mathcal{S}=\{u\in C^{2,\alpha,\frac{-2}{p-1}}(\mathbf{B}_{0}) : u \text{ satisfies } (1)\},$$

where for an integer $j \geq 0, \alpha \in (0,1)$, and $\nu \in \mathbf{R}$, let $C^{j,\alpha,\nu}(\mathbf{B}_0)$ be a weighted Hölder space with a weight parameter ν, defined by

$$C^{j,\alpha,\nu}(\mathbf{B}_0) = \left\{u \in C^{j,\alpha}_{loc}(\mathbf{B}_0) : ||u||_{j,\alpha,\nu} < +\infty\right\},$$

where

$$||u||_{j,\alpha,\nu} = \sup_{0<r\leq 1/2} \left(\sum_{\beta\in\{0,\alpha\}} \sum_{k=0}^{j} r^{k+\beta-\nu} ||\nabla^k u||_{(\beta),B_{2r}\setminus B_r} \right) + ||u||_{j,\alpha,B_1\setminus B_{\frac{1}{2}}}$$

is a norm of $C^{j,\alpha,\nu}(\mathbf{B}_0)$.

Note that for $p > \frac{n}{n-2}$, the set $\mathcal{S}$ is not empty and especially there is a singular radial solution u_0 in $\mathcal{S}$ of the form

$$u_0(x) = C_{n,p}|x|^{\frac{-2}{p-1}} \quad (x \in \mathbf{B}_0), \tag{3}$$

where

$$C_{n,p} = \left\{ \left(\frac{2}{p-1}\right)\left(n - \frac{2p}{p-1}\right)\right\}^{\frac{1}{p-1}}.$$

Now we are interested in the structure of the set $\mathcal{S}$, for example we want to know whether $\mathcal{S}$ has a manifold structure or not, but we cannot find the answer until now. For related problems on the moduli space of solutions of conformal scalar curvature equation with prescribed isolated singularities, see the recent study of [MPU]. Here, utilizing the spherical symmetry of u_0, we prove that, *near u_0, locally $\mathcal{S}$ is a smooth Banach manifold.*

2 Analysis of the Jacobi operator

For $\alpha \in (0,1)$ and $\nu > \frac{-2}{p-1}$, define

$$N(u_0 + v) = \Delta(u_0 + v) + (u_0 + v)^p \tag{4}$$

for v in the small neighborhood $\mathcal{U}$ of 0 in $C^{2,\alpha,\nu}(\mathbf{B}_0)$. N is a smooth map from $\{u_0\} + \mathcal{U}$ to $C^{0,\alpha,\nu-2}(\mathbf{B}_0)$ and the linearized operator (*the Jacobi operator*) about u_0 is given by

$$J_{u_0}\kappa = \left.\frac{d}{dt}\right|_{t=0} N(u_0 + t\kappa) = \Delta\kappa + pu_0^{p-1} \cdot \kappa \tag{5}$$

for $\kappa \in C^{2,\alpha,\nu}(\mathbf{B}_0)$.

Using the polar coordinates $x = r\theta \quad (r = |x|, \theta \in S^{n-1})$ on $\mathbf{B}_0$ and the explicit form of u_0, we can write J_{u_0} as

$$J_{u_0} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}} + \frac{A_{n,p}}{r^2} \tag{6}$$

where $\Delta_{S^{n-1}}$ is the Laplace operator on S^{n-1}, and

$$A_{n,p} = p(C_{n,p})^{p-1} = (\frac{2p}{p-1})(n - \frac{2p}{p-1}). \tag{7}$$

Let $\{\lambda_j\} : 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots \to +\infty$ be the eigenvalues of $\Delta_{S^{n-1}}$ (counting multiplicity) and $\{\eta_j\}$ be the corresponding L^2-normalized eigenfunctions.

As in [CHS][NS], we separate variables and write

$$\begin{aligned} \kappa(r\theta) &= \sum_{j=0}^{\infty} \kappa_j(r)\eta_j(\theta), \quad \kappa_j(r) = \langle \kappa(r\cdot), \eta_j(\cdot)\rangle_{L^2(S^{n-1})}, \\ f(r\theta) &= \sum_{j=0}^{\infty} f_j(r)\eta_j(\theta), \quad f_j(r) = \langle f(r\cdot), \eta_j(\cdot)\rangle_{L^2(S^{n-1})} \end{aligned}$$

for $\kappa \in C^{2,\alpha,\nu}(\mathbf{B}_0)$ and $f \in C^{0,\alpha,\nu-2}(\mathbf{B}_0)$.

Then, formally the equation $J_{u_0}\kappa = f$ is equivalent to

$$\kappa_j''(r) + \frac{n-1}{r}\kappa_j'(r) - \frac{\lambda_j - A_{n,p}}{r^2}\kappa_j(r) = f_j(r), \quad j = 0, 1, 2\cdots \tag{8}$$

which are inhomogeneous Euler ODE's.

Let for $j = 0, 1, 2, \cdots$,

$$\gamma_j(\pm) = \frac{2-n}{2} \pm \sqrt{\frac{(n-2)^2}{4} + \lambda_j - A_{n,p}}, \tag{9}$$

be the indicial roots of the characteristic equation $x^2 + (n-2)x - (\lambda_j - A_{n,p}) = 0$, and let

$$D_j = \frac{(n-2)^2}{4} + \lambda_j - A_{n,p} \tag{10}$$

be the discriminant.

Note that $\lambda_1 = n - 1$ and

$$D_1 = \frac{(n-2)^2}{4} + (n-1) - (\frac{2p}{p-1})(n - \frac{2p}{p-1}) = (\frac{n}{2} - \frac{2p}{p-1})^2 \geq 0,$$

so $\gamma_j(\pm) \in \mathbf{R}$ for $j \geq 1$.

The general solution g_j of the homogeneous equation associated with (8) is

$$g_j(r) = \begin{cases} a_j Re(r^{\gamma_j(+)}) + b_j Im(r^{\gamma_j(-)}), & j \in \{j : D_j < 0\}; \\ a_j r^{\frac{2-n}{2}} + b_j r^{\frac{2-n}{2}} \log r, & j \in \{j : D_j = 0\}; \\ a_j(r^{\gamma_j(+)}) + b_j(r^{\gamma_j(-)}), & j \in \{j : D_j > 0\}, \end{cases} \tag{11}$$

where, a_j, b_j are constants.

A particular solution $F_j(r)$ of (8) is also known and explicitly given by

$$F_j(r) = \begin{cases} Re\left[r^{\gamma_j(+)} \int_0^r \tau^{1-n-2\gamma_j(+)} \int_0^\tau s^{n-1+\gamma_j(+)} f_j(s) ds d\tau\right], & Re\gamma_j(+) < \nu, \\ Re\left[r^{\gamma_j(+)} \int_1^r \tau^{1-n-2\gamma_j(+)} \int_0^\tau s^{n-1+\gamma_j(+)} f_j(s) ds d\tau\right], & Re\gamma_j(+) > \nu. \end{cases} \tag{12}$$

We note that when $\nu \in \mathbf{R}$ satisfies

$$(13) \qquad \nu \notin \{Re\gamma_j(+) : j = 0,1,2,\cdots\} \quad \text{and} \quad \nu > Re\gamma_0(-),$$

then the functions $\{F_j(r)\}$ are well defined and satisfy the estimate $|F_j(r)| \leq Cr^{\nu}$.

Thus the solution κ of $J_{u_0}\kappa = f$ can be written as

$$(14) \qquad \kappa(r\theta) = \sum_{j=0}^{\infty} g_j(r)\eta_j(\theta) + \sum_{j=0}^{\infty} F_j(r)\eta_j(\theta)$$

for $r\theta = x \in \mathbf{B}_0$.

Let

$$(15) \qquad K_\nu(J_{u_0}) = \{\kappa \in C^{2,\alpha,\nu}(\mathbf{B}_0) : J_{u_0}\kappa = 0\}$$

be the kernel of J_{u_0} in $C^{2,\alpha,\nu}(\mathbf{B}_0)$, that is the set of Jacobi fields.

Then we can see, as in [HM],

Lemma 1 *If $\nu \notin \{Re\gamma_j(+) : j = 0,1,2,\cdots\}$ and $\nu > \frac{2-n}{2}$, then*

$$K_\nu(J_{u_0}) = \{\kappa \in C^{2,\alpha,\nu}(\mathbf{B}_0) : \kappa(r\theta) = \sum_{\gamma_j(+)>\nu} a_j r^{\gamma_j(+)}\eta_j(\theta)\}$$

for some constants $\{a_j\}$.

From now on, we denote

$$(16) \qquad p^* = \max\left(\frac{-2}{p-1}, \frac{2-n}{2}\right)$$

and

$$(17) \qquad L = \min\{j = 0,1,2,\cdots : p^* < Re\gamma_j(+)\}.$$

Note that if $\gamma_0(+) \notin \mathbf{R}$, then $Re\gamma_0(+) = \frac{2-n}{2} \leq p^*$, so $L \neq 0$ and we have always $\gamma_L(+) \in \mathbf{R}$.

We fix $\nu \in \mathbf{R}$ such that

$$(18) \qquad p^* < \nu < \gamma_L(+),$$

so $C^{2,\alpha,\nu}(\mathbf{B}_0) \subset C^{2,\alpha,\frac{-2}{p-1}}(\mathbf{B}_0)$ and (13) is satisfied for this ν.

Denote $I_1 = \{0,1,2,\cdots,L\}$ and $I_2 = \{L, L+1,\cdots\}$, then by Lemma 1 we have

$$K_\nu(J_{u_0}) = \{\kappa \in C^{2,\alpha,\nu}(\mathbf{B}_0) : \kappa(r\theta) = \sum_{j\in I_2} a_j r^{\gamma_j(+)}\eta_j(\theta)\}.$$

Define

$$\begin{aligned} C_k^{2,\alpha,\nu}(\mathbf{B}_0) &= \{\xi \in C^{2,\alpha,\nu}(\mathbf{B}_0) : \xi(r\theta) = \sum_{j\in I_k} a_j(r)\eta_j(\theta)\}, \quad k = 1,2, \\ C_k^{2,\alpha}(S^{n-1}) &= \{\psi \in C^{2,\alpha}(S^{n-1}) : \psi(\theta) = \sum_{j\in I_k} a_j\eta_j(\theta)\}, \quad k = 1,2, \end{aligned}$$

where $\{a_j(r)\}, \{a_j\}$ are some functions and constants, and

$$C_*^{2,\alpha,\nu}(\mathbf{B}_0) = \{\kappa \in C^{2,\alpha,\nu}(\mathbf{B}_0) : \kappa|_{S^{n-1}} \in C_1^{2,\alpha}(S^{n-1})\}, \tag{19}$$

that is, $\kappa \in C_*^{2,\alpha,\nu}(\mathbf{B}_0)$ is a function such that $\kappa|_{S^{n-1}}$ is spanned by $\eta_0, \eta_1, \cdots, \eta_{L-1}$.

Let $\Pi_k : C^{2,\alpha}(S^{n-1}) \to C_k^{2,\alpha}(S^{n-1})$ be the projection

$$\Pi_k : \sum_{j=0}^{\infty} a_j\eta_j(\theta) \mapsto \sum_{j\in I_k} a_j\eta_j(\theta), \quad k = 1, 2,$$

then we can write

$$C_*^{2,\alpha,\nu}(\mathbf{B}_0) = \{\kappa \in C^{2,\alpha,\nu}(\mathbf{B}_0) : \Pi_2(\kappa|_{S^{n-1}}) = 0\}.$$

By exploiting the formulae (11)(12)(14), we have

Lemma 2

(a) *For any* $\psi \in C^{2,\alpha}(S^{n-1})$ *and* $f \in C^{0,\alpha,\nu-2}(\mathbf{B}_0)$, *there exists a unique* $\kappa \in C^{2,\alpha,\nu}(\mathbf{B}_0)$ *such that*

$$\begin{cases} J_{u_0}\kappa = f, \\ \Pi_2(\kappa|_{S^{n-1}}) = \Pi_2(\psi) \quad \text{on } S^{n-1}. \end{cases}$$

(b)

$$J_{u_0}|_{C_*^{2,\alpha,\nu}(B_0)} : C_*^{2,\alpha,\nu}(\mathbf{B}_0) \to C^{0,\alpha,\nu-2}(\mathbf{B}_0)$$

is a linear isomorphism.

Proof

(a) See [CHS].

(b) Let $\kappa \in C_*^{2,\alpha,\nu}(\mathbf{B}_0)$ be a solution of $J_{u_0}\kappa = 0$. Then $\kappa \in K_\nu(J_{u_0})$, so by Lemma 1,

$$\kappa(r\theta) = \sum_{j\in I_2} a_j r^{\gamma_j(+)}\eta_j(\theta),$$

and $0 = \Pi_2(\kappa|_{S^{n-1}}) = \sum_{j\in I_2} a_j\eta_j(\theta)$, which implies $a_j = 0$ for all $j \in I_2$. So J_{u_0} is injective.

For any $f \in C^{0,\alpha,\nu-2}(\mathbf{B}_0)$, by (a) for $\psi = 0$ there is a unique solution $\kappa \in C^{2,\alpha,\nu}(\mathbf{B}_0)$ such that

$$\begin{cases} J_{u_0}\kappa = f, \\ \Pi_2(\kappa|_{S^{n-1}}) = 0. \end{cases}$$

So $\kappa \in C_*^{2,\alpha,\nu}(\mathbf{B}_0)$ and J_{u_0} is surjective. □

3 Implicit function theorem argument

Here we describe the local structure of $\mathcal{S}$ near the singular radial solution u_0.

Theorem 1 *For $n \geq 3$, $p > \frac{n}{n-2}$, $\alpha \in (0,1)$, let $u_0(x) = C_{n,p}|x|^{\frac{-2}{p-1}}$ be the singular radial solution of (1), given by (3), and fix $\nu \in (p^*, \gamma_L(+))$, where p^*, L are as in (16),(17).*

Then, there exists a neighborhood U of 0 in $C_2^{2,\alpha}(S^{n-1})$, a neighborhood V of u_0 in $C^{2,\alpha,\frac{-2}{p-1}}(\mathbf{B}_0)$, and a smooth map $F : U \to V$ such that the following holds:

(1) *$F(0) = u_0$,*

(2) *$F(\psi) \in \mathcal{S}$ for any $\psi \in U$,*

(3) *F is an immersion at 0, that is, $DF(0) : C_2^{2,\alpha}(S^{n-1}) \to C^{2,\alpha,\nu}(\mathbf{B}_0) \subset C^{2,\alpha,\frac{-2}{p-1}}(\mathbf{B}_0)$ is a splitting injection.*

(4) *There is an $\varepsilon > 0$ such that any $v \in \mathcal{S} \cap V_\varepsilon = \mathcal{S} \cap \{v \in C^{2,\alpha,\frac{-2}{p-1}}(\mathbf{B}_0) : ||v - u_0||_{2,\alpha,\nu} < \varepsilon\}$ can be written as $v = F(\psi)$ for some $\psi \in U$.*

(5) *U and ε can be chosen so that $\mathcal{S} \cap V_\varepsilon$ is a smooth manifold diffeomorphic to U. Furthermore, the tangent space of $\mathcal{S} \cap V_\varepsilon$ at u_0 is*

$$T_{u_0}(\mathcal{S} \cap V_\varepsilon) = K_\nu(J_{u_0}).$$

Sketch of proof

Any $\psi \in C_2^{2,\alpha}(S^{n-1})$ can be extended to a Jacobi field $\overline{\psi} \in K_\nu(J_{u_0})$ as follows :

$$\psi(\theta) = \sum_{j \in I_2} a_j \eta_j(\theta) \mapsto \overline{\psi}(r\theta) = \sum_{j \in I_2} a_j r^{\gamma_j(+)} \eta_j(\theta) \tag{20}$$

(See Lemma 1).

Consider the map

$$\Psi : C_2^{2,\alpha}(S^{n-1}) \times C_*^{2,\alpha,\nu}(\mathbf{B}_0) \to C^{0,\alpha,\nu-2}(\mathbf{B}_0)$$

defined by

$$\Psi(\psi, \kappa) = N(u_0 + \overline{\psi} + \kappa) = \Delta(u_0 + \overline{\psi} + \kappa) + (u_0 + \overline{\psi} + \kappa)^p. \tag{21}$$

Note that Ψ is well defined for (ψ, κ) in a small neighborhood of $(0,0) \in C_2^{2,\alpha}(S^{n-1}) \times C_*^{2,\alpha,\nu}(\mathbf{B}_0)$, and

$$\begin{aligned}
\Psi(0,0) &= \Delta u_0 + u_0^p = 0, \\
D_2\Psi(0,0) &= DN(u_0) = J_{u_0} : C_*^{2,\alpha,\nu}(\mathbf{B}_0) \to C^{0,\alpha,\nu-2}(\mathbf{B}_0)
\end{aligned}$$

is a linear isomorphism by Lemma 2(b).

So by the implicit function theorem, there are neighborhoods U of 0 in $C_2^{2,\alpha}(S^{n-1})$, W of 0 in $C_*^{2,\alpha,\nu}(\mathbf{B}_0)$ and a smooth map $Q : U \to W$ such that $Q(0) = 0$ and for any $\psi \in U$, $Q(\psi)$ is the unique solution of

$$\Psi(\psi, Q(\psi)) = N(u_0 + \overline{\psi} + Q(\psi)) = 0.$$

Finally we define a smooth map $F : U \subset C_2^{2,\alpha}(S^{n-1}) \to C^{2,\alpha,\frac{-2}{p-1}}(\mathbf{B}_0)$ as

$$F(\psi) = u_0 + \overline{\psi} + Q(\psi). \tag{22}$$

Note for $\psi \in U$ near 0, $\overline{\psi} + Q(\psi)$ is small compared to u_0 in $C^{2,\alpha,\nu}(\mathbf{B}_0) \subset C^{2,\alpha,\frac{-2}{p-1}}(\mathbf{B}_0)$, and $F(\psi)$ has a form that u_0 + (perturbation behaving like r^ν near 0).

Now it is easy to see that F satisfies **(1)(2)**.

To see that F is an immersion at 0, first note that for any $\psi \in C_2^{2,\alpha}(S^{n-1})$,

$$DF(0)\psi = \left.\frac{d}{dt}\right|_{t=0} (u_0 + (\overline{t\psi}) + Q(t\psi)) = \overline{\psi} + DQ(0)\psi \in C^{2,\alpha,\nu}(\mathbf{B}_0),$$

so $\Pi_2\left((DF(0)\psi)|_{S^{n-1}}\right) = \Pi_2(\psi)$ since $DQ(0)\psi \in C_*^{2,\alpha,\nu}(\mathbf{B}_0)$.

Define a map $\Lambda : C^{2,\alpha,\nu}(\mathbf{B}_0) \to C_2^{2,\alpha}(S^{n-1}) \times C_*^{2,\alpha,\nu}(\mathbf{B}_0)$ such that

$$\Lambda(\xi) = \left(\Pi_2\left(\xi|_{S^{n-1}}\right),\ \xi - DF(0)\left(\Pi_2(\xi|_{S^{n-1}})\right)\right).$$

Obviously Λ is injective, and for any $(\psi, \eta) \in C_2^{2,\alpha}(S^{n-1}) \times C_*^{2,\alpha,\nu}(\mathbf{B}_0)$, if we set $\xi = DF(0)\psi + \eta$ then $\xi \in C^{2,\alpha,\nu}(\mathbf{B}_0)$ and $\Lambda(\xi) = (\psi, \eta)$, so Λ is surjective and therefore Λ is a bounded linear isomorphism.

Consider the sequence of mappings

$$C_2^{2,\alpha}(S^{n-1}) \xrightarrow{DF(0)} C^{2,\alpha,\nu}(\mathbf{B}_0) \xrightarrow{\Lambda} C_2^{2,\alpha}(S^{n-1}) \times C_*^{2,\alpha,\nu}(\mathbf{B}_0) \xrightarrow{Pr_1} C_2^{2,\alpha}(S^{n-1}),$$

where Pr_1 is the projection, then we see

$$Pr_1 \circ \Lambda \circ DF(0) = Id|_{C_2^{2,\alpha}(S^{n-1})},$$

so $DF(0)$ is a splitting injection. This proves **(3)**.

To show **(4)**, take $\varepsilon > 0$ sufficiently small so that $N(v) = \Delta v + v^p$ is well defined and $\Pi_2((v - u_0)|_{S^{n-1}}) \in U$ for $v \in V_\varepsilon$. Then given $v \in \mathcal{S} \cap V_\varepsilon$, let $\kappa = v - u_0 - \overline{\psi}$, where $\overline{\psi} \in K_\nu(J_{u_0})$ is a Jacobi field defined by (20) for $\psi = \Pi_2((v - u_0)|_{S^{n-1}})$. Since $\Pi_2(\kappa|_{S^{n-1}}) = 0$, we have $\kappa \in C_*^{2,\alpha,\nu}(\mathbf{B}_0)$.

Now $v = u_0 + \kappa + \overline{\psi} \in \mathcal{S}$ implies $\Psi(\psi, \kappa) = 0$, then by the uniqueness of $Q(\psi)$ for $\psi \in U$, we have $\kappa = Q(\psi)$. By the definition of F, we get $v = F(\psi)$, which proves **(4)**.

(5) follows from the theory of immersions between Banach spaces. □

4 An application

Here, following the arguments in previous sections, we give a result about the existence of solutions for a perturbed equation, that are singular only at the origin.

Theorem 2 *For any $\varepsilon > 0, \nu \in (p^*, \gamma_L(+))$, there is a $\delta > 0$ such that if $K \in C^{2,\alpha}(\mathbf{B}_1)$ is a positive function with $||K - 1||_{C^{2,\alpha}(B_1)} \leq \delta$, then there exists a solution $v \in C^{2,\alpha,\frac{-2}{p-1}}(\mathbf{B}_0)$ of*

$$\begin{cases} \Delta v + K(x)v^p = 0 \ \ in \ \mathcal{D}'(\mathbf{B}_1), \\ ||v - u_0||_{2,\alpha,\nu} < \varepsilon. \end{cases}$$

Proof

For $K \in C^{2,\alpha}(\mathbf{B}_1)$ and $u \in C^{2,\alpha,\nu}(\mathbf{B}_0)$ near u_0, denote

$$N(K, u) = \Delta u + K(x)u^p.$$

N is a smooth map to $C^{0,\alpha,\nu-2}(\mathbf{B}_0)$ and if we define a map $\Phi : C^{2,\alpha}(\mathbf{B}_1) \times C_*^{2,\alpha,\nu}(\mathbf{B}_0) \to C^{0,\alpha,\nu-2}(\mathbf{B}_0)$ as

$$\Phi(\eta, \kappa) = N(1 + \eta, u_0 + \kappa) = \Delta(u_0 + \kappa) + (1 + \eta(x))(u_0 + \kappa)^p,$$

then we see $\Phi(0,0) = N(1, u_0) = 0$ and $D_2\Phi(0,0) = J_{u_0} : C_*^{2,\alpha,\nu}(\mathbf{B}_0) \to C^{0,\alpha,\nu-2}(\mathbf{B}_0)$ is a linear isomorphism by Lemma 2(b).

So by the implicit function theorem, we have a neighborhood U of 0 in $C^{2,\alpha}(\mathbf{B}_1)$, V of 0 in $C_*^{2,\alpha,\nu}(\mathbf{B}_0)$, and a smooth map $Q : U \to V$ such that $Q(0) = 0$ and for any $\eta \in U$, $Q(\eta)$ is the unique solution of $\Phi(\eta, Q(\eta)) = 0$. Furthermore for any $\varepsilon > 0$, if $||\eta||_{2,\alpha} \leq \delta$ for sufficiently small δ, we have $||Q(\eta)||_{2,\alpha,\nu} < \varepsilon$ by continuity of Q.

Denote $v = u_0 + Q(\eta)$ where $\eta = K - 1$, then $\Phi(\eta, Q(\eta)) = 0$ implies $\Delta v + K(x)v^p = 0$ in $C^{0,\alpha,\nu-2}(\mathbf{B}_0)$. Now $p > \frac{n}{n-2}$ allows that v extends to the whole ball as a solution in the distribution sense. The proof is completed. □

References

[CGS] L. Caffarelli, B. Gidas, and J. Spruck. *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth,* Comm. Pure Apll. Math. **42**, 1989, pp271-297

[CHS] L. Caffarelli, R. Hardt, and L. Simon. *Minimal surfaces with isolated singularities,* Manuscripta Math. **48**, 1984, pp1-18

[GS] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations,* Comm. Pure Appl. Math. **34**, 1981, pp525-598

[HM] R. Hardt, and L. Mou, *Harmonic maps with fixed singular sets,* J. Geom. Anal. **2**, No.5, 1992, pp445-488

[MPU] R. Mazzeo, D. Pollack, and K. Uhlenbeck. *Moduli spaces of singular Yamabe metrics,* J. Amer. Math. Soc. **9**, 1996, pp303-344

[NS] N. Smale, *An equivariant construction of minimal surfaces with nontrivial singular sets,* Indiana. Univ. Math. J. **40**, 1991, pp595-616

[PL] P. Lions, *Isolated singularities in semilinear problems,* J. Diff. Eqn. **38**, No.3, 1980, pp441-450